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by G. A. Krasinskiy

From: *Byulleten' Instituta Teoreticheskoy Astronomii,*
Volume 9, No. 2, 1963.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1967

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Translation of "Zamknutyye trayektorii soudareniya v ploskoy krugovoy ogranichennoy zadache trekh tel." Byulleten' Instituta Teoreticheskoy Astronomii, Vol. 9, No. 2, pp. 154-167, 1963.

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CLOSED COLLISION PATHS IN A PLANE CIRCULAR RESTRICTED THREE-BODY PROBLEM /154*

G.A.Krasinskiy

The paths of double collision with a greater mass in the plane circular restricted three-body problem are studied. The existence of certain types of paths both symmetric and asymmetric with respect to the x-axis is shown by the small-parameter method. The method of constructing these paths is given.

In this paper, we study the paths of double collision with a large body in the two-dimensional circular restricted three-body problem. These paths are of importance for astronautics, as orbits along which flights around the moon and return to the earth are possible utilizing only the gravitational forces of both earth and moon. Mathematically, the problem reduces to solving a system of equations of the two-dimensional circular restricted problem

$$\left.\begin{aligned}\frac{d^2x}{dt^2}-\frac{2dy}{dt}-x&=\frac{\partial U}{\partial x},\\ \frac{d^2y}{dt^2}+\frac{2dx}{dt}-y&=\frac{\partial U}{\partial y},\end{aligned}\right\} \tag{1}$$

where $U = \dfrac{1-\mu}{\sqrt{(x+\mu)^2+y^2}} + \dfrac{\mu}{\sqrt{(x+\mu-1)^2+y^2}}$, with boundary conditions coinciding with the position of greater masses

$$\begin{aligned}x(0)&=x(\tau)=-\mu,\\ y(0)&=y(\tau)=0.\end{aligned} \tag{2}$$

Since, in the solution being sought, the right-hand side of eq.(1) becomes infinite, the Levi-Civita regularizing variables are employed.

The solution of the boundary-value problem [eqs.(1), (2)] is sought in the form of power series of the small parameter μ. For $\mu = 0$, the solution of the corresponding two-body problem depends on one arbitrary parameter ω (longitude of perihelion), as a consequence of which the construction of the successive approximations encounters the same difficulty as in the theory of periodic solutions, namely, the vanishing of some determinant. For this reason the solution of the problem (1), (2) will not be valid for all values of ω but only for those that satisfy a certain equation. Having constructed series formally satisfying

* Numbers given in the margin indicate pagination in the original foreign text.

eqs.(1), (2), we can easily show their convergence by methods analogous to those used in the theory of periodic solutions.

1. Limiting Case: $\mu = 0$

Let us change, in eqs.(1), (2), to Levi-Civita regularizing variables by means of the formulas (Wintner, 1941):

$$x+\mu=p^2-q^2, \quad y=2pq, \quad dt=4(p^2+q^2)\,ds.$$

With the new variables, the equations of motion will have the form /155

$$\left.\begin{aligned} \ddot{p}-8(p^2+q^2)\dot{q}&=\frac{\partial U}{\partial p}, \\ \ddot{q}+8(p^2+q^2)\dot{p}&=\frac{\partial U}{\partial q}, \end{aligned}\right\} \tag{3}$$

where

$$U=4(p^2+q^2)\left[\frac{\mu^2}{2}-\mu(p^2-q^2)+\frac{(p^2+q^2)^2}{2}+\frac{1-\mu}{p^2+q^2}+\frac{\mu}{\sqrt{1-2(p^2-q^2)+(p^2+q^2)^2}}-\frac{c}{2}\right].$$

Here we must stipulate that the equality $\frac{1}{2}(\dot{p}^2 + \dot{q}^2) - U = \frac{h}{2} = 0$ be satisfied in solving the system (3). The boundary conditions with the new variables read as follows:

$$p(0)=p(\sigma)=q(0)=q(\sigma)=0, \tag{4}$$

where σ is the fly-around time. Equation (3) can be written in Hamiltonian form

$$\left.\begin{aligned} \dot{p}&=\frac{\partial H}{\partial P}, & \dot{q}&=\frac{\partial H}{\partial Q}, \\ \dot{P}&=-\frac{\partial H}{\partial p}, & \dot{Q}&=-\frac{\partial H}{\partial q}, \end{aligned}\right\} \tag{5}$$

where

$$H=\frac{1}{2}(P^2+Q^2)+2(qP-pQ)(p^2+q^2)+4(p^2+q^2)\times$$

$$\times\left[\frac{(1-\mu)}{p^2+q^2}+\frac{f\mu}{\sqrt{1-2(p^2-q^2)+(p^2+q^2)^2}}-\frac{c}{2}\right]. \tag{6}$$

We will denote p, q, P, Q respectively by x_1, x_2, x_3, x_4, and write eq.(5) as

$$\dot{x}_j=X_j(x_1, x_2, x_3, x_4, c, \mu) \qquad j=1, 2, 3, 4, \tag{7}$$

$$x_1(0)=x_1(\sigma)=x_2(0)=x_2(\sigma)=0. \tag{8}$$

Let us find the solution of the problem (3) and (4) for $\mu = 0$. Here, eqs.(3), for $\mu = 0$, have the form

$$\begin{rcases} \ddot{p} - 8(p^2+q^2)\dot{q} = 12(p^2+q^2)^2 p - 4cp, \\ \ddot{q} + 8(p^2+q^2)\dot{p} = 12(p^2+q^2)^2 q - 4cq. \end{rcases} \tag{9}$$

This system has two integrals

$$\begin{rcases} \frac{1}{2}(\dot{p}^2+\dot{q}^2) = 2(p^2+q^2)^3 - 2c(p^2+q^2) + 4 + \frac{h}{2}, \\ \dot{p}q - q\dot{p} - 2(p^2+q^2) = b. \end{rcases} \tag{10}$$

Let us introduce the polar coordinates

$$p = \rho\sin\theta, \quad q = \rho\cos\theta. \tag{11}$$

Substituting eq.(11) into eq.(10), we obtain

$$\rho^2\dot{\theta} - 2\rho^4 = b, \tag{12}$$

$$\dot{\rho}^2 + \rho^2\dot{\theta}^2 = 4\rho^6 - 4c\rho^2 + 8 + h. \tag{13}$$

Since $\rho = 0$ for $s = 0$, it follows from eq.(12) that $b = 0$. Substituting $\dot{\theta} = 2\rho^2$ into eq.(13) will give

$$s = \int_0^\rho \frac{d\rho}{\sqrt{-4c\rho^2 + 8 + h}}.$$

Let us substitute the variables $\rho = \sqrt{\frac{2}{c} + \frac{h}{4c}}\sin E$ /156

$$s = \frac{\sqrt{\frac{1}{c}\left(2 + \frac{h}{4}\right)}}{\sqrt{8+h}} \int_0^E \frac{\cos E}{\sqrt{1-\sin^2 E}}\, dE = \frac{1}{\sqrt{4c}} E,$$

but $c = \frac{1}{a}$, where a is the semimajor axis; consequently,

$$\rho = \sqrt{\frac{1}{c}\left(2 + \frac{h}{4}\right)}\sin 2ans \tag{14}$$

where n is the mean diurnal motion.

Making use of the equality $\dot{\theta} = 2\rho^2$, we find θ

$$\theta = \frac{8+h}{16}\left(4as - \frac{\sin 4ans}{n}\right) + \omega. \tag{15}$$

Thus, the solution of the boundary-value problem (3) and (4), for $\mu = 0$, depends on two arbitrary constants ω and h. For $h = 0$, we obtain the solution of the two-body problem satisfying the boundary condition (4).

We see from eq.(14) that the fly-around time σ is connected to the semimajor axis a by the formula $\sigma = \frac{\pi}{2an} k$, where k is some integer.

2. Determination of First-Order Perturbations

To determine the perturbations we must find the general solution of a set of homogeneous variational equations

$$\left.\begin{aligned}&\delta\ddot{p}-8r\delta\dot{q}-16(p\delta p+q\delta q)(\dot{q}+3pr)-(12r^2-4c)\delta p=0,\\&\delta\ddot{q}+8r\delta\dot{p}-16(p\delta p+q\delta q)(-\dot{p}+3qr)-(12r^2-4c)\delta q=0,\end{aligned}\right\} \tag{16}$$

where $r = p^2 + q^2$, p, q are calculated on the basis of eqs.(11), (14), (15) for h = 0. The system (16) has two trivial solutions

$$\left.\begin{aligned}\delta p_1&=q,\\ \delta q_1&=-p,\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}\delta p_2&=\dot{p},\\ \delta q_2&=\dot{q}.\end{aligned}\right\} \tag{17}$$

Differentiating p and q with respect to h and making use of eqs.(11), (14), (15), we obtain

$$\frac{\partial p}{\partial h}=\frac{\partial \rho}{\partial h}\sin\theta+\rho\cos\theta\frac{\partial\theta}{\partial h}=\frac{\sqrt{2a}}{16}\sin 2ans\sin\left(\frac{t}{2}+\omega\right)+\frac{\sqrt{2a}}{16}t\sin 2ans\omega s\left(\frac{t}{2}+\omega\right),$$

where

$$t=4as-\frac{\sin 4ans}{n}.$$

Analogously,

$$\frac{\partial q}{\partial h}=\frac{\sqrt{2a}}{16}(q-tp).$$

Consequently, the third particular solution of the system (13) will be

$$\left.\begin{aligned}\delta p_3&=p+tq,\\ \delta q_3&=q-tp.\end{aligned}\right\} \tag{18}$$

By a direct check we can convince ourselves that the fourth solution is /157

$$\left.\begin{aligned}\delta p_4&=q\cot 2ans+\frac{2}{n}\dot{p}s-\frac{qt}{n},\\ \delta q_4&=-p\cot 2ans+\frac{2}{n}\dot{q}s+\frac{pt}{n}.\end{aligned}\right\} \tag{19}$$

It is easy to demonstrate that these solutions are linearly independent. Let us seek the solution of the problem (7), (8) in the form of a power series of μ

$$x_j=\sum_{i=0}^{\infty}x_j^{(i)}\mu^i. \tag{20}$$

Let us also consider the Jacobi constant c on the r.h.s. of eqs.(7), which can be determined in the form of a series

$$c = c_0 + \sum_{1=1}^{\infty} c_i \mu^i, \quad c_0 = \frac{1}{\alpha} = \frac{\pi^2 k^2}{4\sigma^2} \quad k = 1, 2, 3 \ldots$$

Let us set $\mu = 0$. The functions $x_1^{(0)} = p$, $x_2^{(0)}$ are calculated, as shown above, from eqs.(11), (14), (15), while the functions $x_3^{(0)}$, $x_4^{(0)}$, as follows from eq.(5), are obtained over the formulas

$$x_3^{(0)} = P = x_1^{(0)} - 2\dot{x}_2^{(0)} r, \quad x_4^{(0)} = Q = \dot{x}_2^{(0)} + 2x_1^{(0)} r. \tag{21}$$

The functions $x_j^{(i)}$ $(i > 0)$ satisfy the system of linear differential equations

$$\frac{dx_j^k}{ds} = \sum_{\alpha=1}^{4} p_{j\alpha} x_\alpha^{(i)} + F_j^{(i)} \left(x^{(0)} \ldots x^{(i-1)}, \ c_0 \ldots c_{i-1}\right) + \varphi_j c_i, \tag{22}$$

where

$$p_{j\alpha} = \frac{\partial X_j \left(x_1^{(0)} \ldots x_4^{(0)}, \ c_0, \ 0\right)}{\partial x_\alpha^{(0)}},$$

$$\varphi_1 = \varphi_2 = 0, \quad \varphi_3 = -4x_1^{(0)}, \quad \varphi_4 = -4x_2^{(0)}, \tag{23}$$

in particular, $F_j^{(1)} = \left.\frac{\partial X_j (x_1^{(0)} \ldots x_4^{(0)}, \ c_0, \ \mu)}{\partial \mu}\right|_{\mu=0}$. The functions $x_j^{(i)}$ should also satisfy the boundary conditions

$$x_1^{(i)}(0) = x_2^{(i)}(0) = x_1^{(i)}(\sigma) = x_2^{(i)}(\sigma) = 0. \tag{24}$$

Knowing the solution of the variational system (16), we easily find all linearly independent solutions of the homogeneous system

$$\frac{dy_j}{ds} = \sum_{\alpha=1}^{4} p_{j\alpha} y_\alpha. \tag{25}$$

In this case,

$$\left.\begin{aligned} y_{1i} &= \delta p_i, \quad y_{2i} = \delta q_i, \\ y_{3i} &= \delta \dot{p}_i - 2\delta q_i r - 2q\delta r_i, \\ y_{4i} &= \delta \dot{q}_i + 2\delta p_i r + 2p\delta r_i. \end{aligned}\right\} \tag{26}$$

The last two expressions are obtained by variation of eq.(21).

Let us now find $x_j^{(1)}$. For this, we must find the particular solution of the system (22) for $i = 1$. Using the variational method of arbitrary parameters, we will obtain the particular solution in the form

$$x_k = \sum_{i=1}^{4} \alpha_i y_{ki}. \tag{27}$$

To find α_i, we derived the system /158

$$\sum_{i=1}^{4} \dot{\alpha}_i y_{ki} = F_k^{(1)} + c_1\varphi_k \qquad k = 1, 2, 3, 4, \tag{28}$$

where

$$F_1^{(1)} = 0, \quad F_2^{(1)} = 0,$$

$$F_3^{(1)} = -16x_1^3 + \frac{8x_1}{r_2} + \frac{8rx_1 - 8r^2x_1}{r_2^3},$$

$$F_4^{(1)} = 16x_2^3 + \frac{8x^2}{r_2} + \frac{-8rx_2 - 8r^2x_2}{r_2^3}.$$

Let

$$z_{1i} = -y_{3i}, \quad z_{2i} = -y_{4i}, \quad z_{3i} = y_{1i}, \quad z_{4i} = y_{2i}. \tag{29}$$

It is known that, for canonical systems, the quantities z_{ki} form the matrix of the solutions for the system of equations conjugate to the system (25) (Malkin, 1956), i.e., for equations

$$\frac{dz_j}{ds} = -\sum_{\alpha=1}^{4} p_{\alpha j} z_\alpha. \tag{30}$$

The basic property of conjugate systems is that the expression $A_{ij} = \sum_k z_{ki} y_{kj}$ does not depend on time (Malkin, 1956). For canonical equations, this property is equivalent to the time-invariance of the Lagrange brackets A_{ij} (Subbotin, 1937). It is obvious from the determination that $A_{ij} = -A_{ji}$. Let us then multiply eq.(28) by z_{kj} and carry out the summation for k from 1 to 4. The obtained system will be equivalent to the initial system, but with constant coefficients for $\dot{\alpha}_i$:

$$\sum_{i=1}^{4} \dot{\alpha}_i A_{ji} = f_j, \tag{31}$$

where

$$f_j = \sum_{k=1}^{4} \left(F_k^{(1)} + c_1\varphi_k\right) z_{kj}.$$

Since the coefficients A_{ij} are independent of time, we will set s = 0 in calculating them.

Making use of eqs.(11), (14), (15), it is easy to calculate the matrix of the system (28) for s = 0. This yields the following matrix:

$$\begin{Vmatrix} 0, & \sqrt{8}\sin\omega, & 0, & \sqrt{2a}\cos\omega \\ 0, & \sqrt{8}\cos\omega, & 0, & -\sqrt{2a}\sin\omega \\ \sqrt{8}\cos\omega, & 0, & \sqrt{8}\sin\omega, & 0 \\ -\sqrt{8}\sin\omega, & 0, & \sqrt{8}\cos\omega, & 0 \end{Vmatrix}$$

Having formed the matrix $\|A_{ji}\|$, we obtain

$$\|A_{ji}\| = \begin{Vmatrix} 0 & 0 & 0 & -4\sqrt{a} \\ 0 & 0 & 8 & 0 \\ 0 & -8 & 0 & 0 \\ 4\sqrt{a} & 0 & 0 & 0 \end{Vmatrix}. \tag{32}$$

Thus, we have the following system: /159

$$\left.\begin{aligned} -4\sqrt{a}\dot{\alpha}_4 &= x_2 f_3 - x_1 f_4, \\ 8\dot{\alpha}_3 &= \dot{x}_1 f_3 + \dot{x}_2 f_4, \\ -8\dot{\alpha}_2 &= (x_1 + t\dot{x}_2) f_3 + (x_2 - t x_1) f_4, \\ 4\sqrt{a}\dot{\alpha}_1 &= \left(x_2 \cot 2ans + \frac{2}{n}\dot{x}_1 s - \frac{x_2 t}{n}\right) f_3 + \left(-x_1 \cot 2ans + \frac{2}{n}\dot{x}_2 s + \frac{x_1 t}{n}\right) f_4. \end{aligned}\right\} \tag{33}$$

Consequently,

$$x_k^{(1)} = \sum_{i=1}^{4} \left(\alpha_i + a_i^{(1)}\right) y_{ki},$$

where α_i are calculated by eqs.(33) and $a_i^{(1)}$ are arbitrary constants. We will arrange these constants such that the equalities

$$x_1^{(1)}(0) = x_2^{(1)}(0) = x_1^{(1)}(\sigma) = x_2^{(1)}(\sigma) = 0. \tag{34}$$

are satisfied.

If, in eq.(33), we take the integration limits from 0 to s, then, for s = = 0, all α_i = 0. Having set s = 0, we have

$$x_1^{(1)}(0) = a_2^{(1)}\sqrt{8}\sin\omega + a_4^{(1)}\sqrt{2a}\cos\omega = 0,$$

$$x_2^{(1)}(0) = a_2^{(1)}\sqrt{8}\cos\omega - a_4^{(1)}\sqrt{2a}\sin\omega = 0.$$

Hence,

$$a_2^{(1)} = a_4^{(1)} = 0$$

Let us set s = σ

$$\left.\begin{aligned} -\alpha_2(\sigma)\sqrt{8}\sin\left(\omega + \frac{\pi}{2n}k\right) - \alpha_4(\sigma)\sqrt{2a}\cos\left(\omega + \frac{\pi}{2n}k\right) = 0, \\ -\alpha_2(\sigma)\sqrt{8}\cos\left(\omega + \frac{\pi}{2n}k\right) + \alpha_4(\sigma)\sqrt{2a}\sin\left(\omega + \frac{\pi}{2n}k\right) = 0 \end{aligned}\right\} \tag{35}$$

or $\alpha_2(\sigma) = \alpha_4(\sigma) = 0$. In this case $a_1^{(1)}$, $a_3^{(1)}$ remain indeterminate.

Thus, for the system of equations (22) to have a solution satisfying the conditions (24) for i = 1, it is necessary and sufficient that the two following conditions be satisfied:

$$P_1(\omega, h) = \int_0^\sigma [x_2 f_3^{(1)} - x_1 f_4^{(1)}]\, ds = \int_0^\sigma \left[-16x_1^3 + \frac{8x_1}{r_2} + \frac{8rx_1 - 8r^2x_1}{r_2^3} - 4c_1x_1 \right] x_2 ds -$$

$$- \int_0^\sigma \left[16x_2^3 + \frac{8x_2}{r_2} + \frac{-8rx_2 - 8r^2x_2}{r_2^3} - 4c_1x_2 \right] x_1 ds = 16 \int_0^\sigma x_1 x_2 r \left(\frac{1}{r_2^3} - 1 \right) ds = 0, \tag{36}$$

$$P_2(\omega, h_1 c_1) = \int_0^\sigma [(x_1 + tx_2) f_3^{(1)} + (x_2 - tx_1) f_4^{(1)}]\, ds =$$

$$= \int_0^\sigma 4r \left[4(x_2^2 - x_1^2) + \frac{2}{r_2} + \frac{2(x_1^2 - x_2^2) - 2r^2}{r_2^3} - c_1 \right] ds + 16 \int_0^\sigma x_1 x_2 r t \cdot \left(\frac{1}{r_2^3} - 1 \right) ds = 0. \tag{37}$$

In these equations, let us set h = 0 and return to the variables x, y, t. We obtain

$$\int_0^\tau y \left(\frac{1}{r_2^3} - 1 \right) dt = 0, \tag{38}$$

$$c_1 \tau + \int_0^\tau \left[4x - \frac{2}{r_2} - \frac{2(r^2 - x)}{r_2^3} \right] ds - 2 \int_0^\tau y \left(\frac{1}{r_2^3} - 1 \right) t\, dt = 0. \tag{39}$$

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From eq.(38) we find ω; substituting this value into eq.(39) we obtain the value of c_1, which is the correction to Jacobi's constant.

3. Construction of the Formal Solution

On fulfilling the conditions (38), (39) the solution of eqs.(22), for i = = 1, which satisfies the conditions (24) depends on two arbitrary constants $a_1^{(1)}$, $a_3^{(1)}$, which remain indeterminate. These arbitrary constants are determined from the conditions of solvability of the boundary-value problem for equations of the second approximation. The value of c_2 is determined simultaneously. The arbitrary constants arising in this case are determined from the conditions of solvability of the boundary-value problem for equations of the third approximation, and so on. We will show in greater detail how this is done. The following transformations are completely analogous to those used in the theory of periodic solutions (Malkin, 1956). Let us assume that we determined all functions $x_j^{(i)}$ (s = 1 ... 4) up to the k-th order inclusive, that they satisfy eqs.(22) and the boundary conditions (24), and that we had determined all constants c_i, i = 1, ..., k.

Let us then enumerate the solution of variational equations, such that $y_{j1} = \frac{\partial x_j}{\partial \omega}$, $y_{j2} = \frac{\partial x_j}{\partial h}$. In this case, $y_{j1}(0) = y_{j1}(\sigma) = y_{j2}(0) = y_{j2}(\sigma) =$ $= 0$ (j = 1, 2). The functions $x_j^{(i)}$ will have the form

$$x_j^{(i)} = a_1^{(i)} y_{j1} + a_2^{(i)} y_{j2} + \bar{x}_j^{(i)}, \tag{40}$$

where $\bar{x}_j^{(i)}$ is the particular solution satisfying the boundary conditions, while $a_1^{(i)}$, $a_2^{(i)}$ are certain constants. The quantities $a_1^{(i)}$, $a_2^{(i)}$ will be considered as already determined for $i = 1 \ldots k - 1$, while $a_1^{(k)}$, $a_2^{(k)}$ are still to be determined from the solvability conditions of the boundary-value problem for equations of the $(k + 1)$-th approximation. The functions $F_j^{(k+1)}$ have the following structure:

$$\begin{aligned} F_s^{(k+1)} = \frac{1}{2} \sum_{j=1}^{2} \sum_{\alpha,\beta=1}^{4} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta}\right) \left(x_\alpha^{(1)} y_{\beta j} + x_\beta^{(1)} y_{\alpha j}\right) a_j^{(k)} + \\ + \sum_{j=1}^{2} \sum_{\alpha=1}^{4} \left[\left(\frac{\partial F_s^{(1)}}{\partial x_\alpha}\right) + c_1 \left(\frac{\partial \varphi_s}{\partial x_\alpha}\right)\right] y_{\alpha j} a_j^{(k)} + R_s^{(k+1)}, \end{aligned} \tag{41}$$

where the brackets denote that the derivatives are taken for values of the parameters $h = 0$, $\omega = \omega^*$ for $\mu = 0$, and where ω^*, c_1 satisfy eqs.(38) and (39). Here, $R_s^{(k+1)}$ are known functions not depending on $a_j^{(k)}$, c_{k+1}. Let us transform these expressions

$$F_s^{(k+1)} = \sum_{j=1}^{2} \sum_{\alpha,\beta=1}^{4} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta}\right) x_\alpha^{(1)} y_{\beta j} a_j^{(k)} + \sum_{j=1}^{2} \sum_{\alpha=1}^{4} \left[\left(\frac{\partial F_s^{(1)}}{\partial x_\alpha}\right) + c_1 \left(\frac{\partial \varphi_s}{\partial x_\alpha}\right)\right] y_{\alpha j} a_j^{(k)} + R_s^{(k+1)}.$$

We will denote the parameters ω, h by h_1, h_2 respectively; then $y_{\beta j} = \frac{\partial x_\beta}{\partial h_j}$ $(j = 1, 2)$, and $F_s^{(k+1)}$ will take the form

$$F_s^{(k+1)} = \sum_{j=1}^{2} \sum_{\alpha,\beta=1}^{4} \left[\frac{\partial p_{s\alpha}}{\partial h_j} x_\alpha^{(1)} + \frac{\partial \left(F_s^{(1)} + c_1 \varphi_s\right)}{\partial h_j}\right] a_j^{(k)} + R_s^{(k+1)}. \tag{42}$$

For $k > 1$, $F_s^{(k+1)}$ are linear relative to $a_j^{(k)}$ but for $k = 1$ there will also be quadratic terms since the constants $a_j^{(k)}$ enter $x_\alpha^{(1)}$

$$F_s^{(2)} = \sum_{\alpha=1}^{4} \sum_{i,j=1}^{2} \frac{\partial p_{s\alpha}}{\partial h_j} y_{\alpha i} a_i^{(1)} a_j^{(1)} + \sum_{j=1}^{2} \left[\sum \frac{\partial p_{s\alpha}}{\partial h_j} x_\alpha^{(1)} + \frac{\partial (F_s + c_1 \varphi_s)}{\partial h_j}\right] a_j^{(1)} + R_s^{(2)}. \tag{43}$$

Let us now calculate the expression /161

$$K_n^{(k+1)} = \int_0^\sigma \sum_{\beta=1}^{4} \left(F_\beta^{(k+1)} + c_{k+1} \varphi_\beta + R^{(k+1)}\right) z_{\beta n} ds \quad (n = 1, 2), \tag{44}$$

where z_{j1}, z_{j2} are solutions of the conjugate system corresponding to the solutions of y_{j1}, y_{j2} of the variational system. In the same manner as used in

Section 2 for k = 0, it is easy to show that equating the quantities $K_n^{(k+1)}$ to zero is a necessary and sufficient condition for the solvability of our boundary-value problem for equations of the (k + 1)-th approximation. We note that $x_\beta^{(1)}$ satisfy the equations

$$\frac{dx_\beta^{(1)}}{ds}=\sum_{\alpha=1}^{4}p_{\beta\alpha}x_\alpha^{(1)}+F_\beta^{(1)}+c_1\varphi_\beta. \tag{45}$$

Let us then differentiate both sides with respect to h_j

$$\frac{d}{ds}\left(\frac{\partial x_\beta^{(1)}}{\partial h_j}\right)=\sum_{\alpha=1}^{4}p_{\beta\alpha}\frac{\partial x_\alpha^{(1)}}{\partial h_j}+\sum_{\alpha=1}^{4}\frac{\partial p_{\beta\alpha}}{\partial h_j}x_\alpha^{(1)}+\frac{\partial\left(F_\beta^{(1)}+c_1\varphi_\beta\right)}{\partial h_j}. \tag{46}$$

Using this equality, we can rewrite the equality $K_n^{(k+1)} = 0$ in the following manner:

$$\left.\begin{aligned}A_{11}a_1^{(k)}+A_{12}a_2^{(k)}+A_{13}c_{k+1}=B_1^{(k)},\\A_{21}a_1^{(k)}+A_{22}a_2^{(k)}+A_{23}c_{k+1}=B_2^{(k)},\end{aligned}\right\} \tag{47}$$

where

$$A_{mn}=\int_0^\sigma\sum_{\beta=1}^{4}\left[\sum_{\alpha=1}^{4}\frac{\partial p_{\beta\alpha}}{\partial h_m}x_\alpha^{(1)}+\frac{\partial\left(F_\beta^{(1)}+c_1\varphi_\beta\right)}{\partial h_m}\right]z_{\beta n}ds=\int_0^\sigma\sum_{\beta=1}^{4}\left[\frac{d}{ds}\left(\frac{\partial x_\beta^{(1)}}{\partial h_m}\right)-\sum_{\alpha=1}^{4}p_{\beta\alpha}\frac{\partial x_\alpha^{(1)}}{\partial h_m}\right]z_{\beta n}ds.$$

If we integrate by parts and take into account that $z_{\beta n}$ satisfy the conjugate system of equations, we obtain

$$A_{mn}=\sum_{\beta=1}^{4}\frac{\partial x_\beta^{(1)}}{\partial h_m}z_{\beta n}\Big|_0^\sigma-\int_0^\sigma\sum_{\beta=1}^{4}\frac{\partial x_\beta^{(1)}}{\partial h_m}\left[\frac{dz_{\beta n}}{ds}+\sum_{\gamma=1}^{4}p_{\gamma\beta}z_{\gamma n}\right]ds=$$

$$=\sum_{\beta=1}^{4}\frac{\partial x_\beta^{(1)}}{\partial h_m}z_{\beta n}\Big|_0^\sigma=\left\{\frac{\partial}{\partial h_m}\left(\sum x_\beta^{(1)}z_{\beta n}\right)-\sum_{\beta=1}^{4}x_\beta^{(1)}\frac{\partial z_{\beta n}}{\partial h_m}\right\}\Big|_0^\sigma,$$

but $x_1^{(1)}(0) = x_1^{(1)}(\sigma) = x_2^{(1)}(0) = x_2^{(1)}(\sigma) = 0$, $z_{\beta n}(0) = z_{\beta n}(\sigma) = 0$ ($\beta = 3, 4$) at any values of the parameters h_1, h_2. Consequently, $\frac{\partial z_{\beta n}(0)}{\partial h_m} = \frac{\partial z_{\beta n}(\sigma)}{\partial h_m} = 0$ $\beta = 3, 4$ m, n = 1, 2. Therefore,

$$\sum_{\beta=1}^{4}x_\beta^{(1)}\frac{\partial z_{\beta n}}{\partial h_m}\Big|_0^\sigma=0,$$

$$A_{mn}=\frac{\partial}{\partial h_m}\left(\sum x_\beta^{(1)}z_{\beta n}\right)\Big|_0^\sigma.$$

On the other hand, we have

$$\frac{d}{ds}\sum_{\beta=1}^{4}x_\beta^{(1)}z_{\beta n}=\sum_{\beta=1}^{4}(p_{\beta\alpha}x_\alpha^{(1)}+F_\beta^{(1)}+c_1\varphi_\beta)z_{\beta n}-\sum_{\beta=1}^{4}x_\beta^{(1)}\sum_{\gamma=1}^{4}p_{\gamma\beta}z_{\gamma n}=\sum_{\beta=1}^{4}(F_\beta^{(1)}+c_1\varphi_\beta)z_{\beta n}$$

and thus also /162

$$A_{mn}=\frac{\partial}{\partial h_m}\int_0^{\sigma}\sum_{\beta=1}^{4}(F_\beta^{(1)}+c_1\varphi_\beta)z_{\beta n}=\frac{\partial P_n(h_1 h_2)}{\partial h_m}. \tag{48}$$

The quantities A_{13} and A_{23} are easily calculated directly

$$A_{13}=\int_0^{\sigma}\left[-4x_1\frac{\partial x_1}{\partial\omega}-4x_2\frac{\partial x_2}{\partial\omega}\right]ds=\int_0^{\sigma}[-4x_1x_2+4x_1x_2]\,ds=0, \tag{49}$$

$$A_{23}=\int_0^{\sigma}\left[-4x_1\frac{\partial x_1}{\partial h}-4x_2\frac{\partial x_2}{\partial h}\right]ds=\int_0^{\sigma}[-4x_1(x_1+tx_2)-4x_2(x_2-tx_1)]\,ds=-4\int_0^{\sigma}r\,ds=-\tau. \tag{50}$$

To eqs.(47) we must add still another equation stipulating that the equality $H = 0$ be satisfied in the solution in question. Since H retains a constant value in the solution, we will set $s = 0$. This yields

$$\frac{1}{2}[\dot{x}_1^2(0)+\dot{x}_2^2(0)]=4(1-\mu).$$

Let us write out the terms for μ^k

$$\dot{x}_1^{(k)}(0)\dot{x}_1(0)+\dot{x}_2^{(k)}\dot{x}_2(0)=B_3^{(k)},$$

where $B_3^{(k)}$ depends only on $\dot{x}_j^{(i)}(0)$, $i < k$.

Substituting here the expression for $x_j^{(k)}$, we obtain

$$a_1^{(k)}[\dot{y}_{11}(0)\dot{x}_1(0)+\dot{y}_{21}(0)\dot{x}_2(0)]+a_2^{(k)}[\dot{y}_{12}(0)\dot{x}_1(0)+\dot{y}_{22}(0)\dot{x}_2(0)]=B_3^{(k)}.$$

The coefficient for $a_1^{(k)}$ is equal to $\dot{x}_2(0)\,\dot{x}_1(0)-\dot{x}_1(0)\,\dot{x}_2(0)=0$, and for $a_2^{(k)}$ it is equal to $\dot{x}_1(0)^2+\dot{x}_2(0)^2=8$.

Finally, we have a system of three linear equations with three unknowns

$$\left.\begin{aligned}\frac{\partial P_1}{\partial\omega}a_1^{(k)}+\frac{\partial P_1}{\partial h}a_2^{(k)}\qquad\qquad &=B_1^{(k)},\\ \frac{\partial P_2}{\partial\omega}a_1^{(k)}+\frac{\partial P_2}{\partial h}a_2^{(k)}-\tau c_{k+1}&=B_2^{(k)},\\ 8a_2^{(k)}\qquad\qquad &=B_3^{(k)}.\end{aligned}\right\} \tag{51}$$

The determinant of this system is $8\tau\frac{\partial P_1}{\partial\omega}$. If ω is the simple root of the equation $P_1(\omega, 0) = 0$, then this determinant is nonzero.

It remains to examine the case $k = 1$ when equations $k_n^{(2)} = 0$ contain terms quadratic with regard to $a_j^{(1)}$. These terms will enter the expressions for $k_n^{(2)}$ in the following form:

$$A_n = \sum_{j, m=1}^{4} A_{jmn} a_j^{(1)} a_m^{(1)}, \qquad A_{jmn} = \int_0^{\sigma} \sum_{\alpha, \beta=1}^{m} \frac{\partial P_{\beta\alpha}}{\partial n_j} y_{\alpha m} z_{\beta n}.$$

However, it is easy to demonstrate that $A_{jmn} = 0$. For this, we must perform the same calculations as for the coefficients A_{mn} and take into account that $y_{\alpha m}$ satisfy the variational equations which are homogeneous, as a consequence of which $A_{jmn} = \frac{\partial}{\partial h_j}\left(\sum y_{\beta m} z_{\beta m}\right)\Big|_0^{\sigma} = \mathrm{const}\Big|_0^{\sigma} = 0$. The linear part of the equations, for $k = 1$, coincides with the linear part of eqs.(51) for $k > 1$.

Thus, the quantities $a_1^{(k)}$, $a_2^{(k)}$, c_{k+1} satisfy a linear system of equations whose homogeneous part does not depend on the number k and has a nonzero determinant.

Thus, having used the above method we obtained series satisfying eq.(7) and the boundary conditions (8), the coefficients of these series being determined uniquely. If we prove the existence of the solution of the problem (7), (8), the convergence of the obtained series will also be proved.

4. Proof of the Existence of a Solution to Problem (7), (8) /163

Let $x_j = x_j(s, \gamma_1, \gamma_2, \gamma_3, \gamma_4, c, \mu)$ be the general solution of the system of equations (7) such that

$$x_j(0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, c, \mu) = \gamma_j. \tag{52}$$

Let us seek the particular solution satisfying the boundary conditions (8). Having substituted $s = 0$, we immediately find that $\gamma_1 = \gamma_2 = 0$.

For $s = \sigma$, we obtain the following system of equations:

$$\left.\begin{aligned} x_1(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = 0, \\ x_2(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = 0. \end{aligned}\right\} \tag{53}$$

Let us stipulate that the equality $H = 0$ must be satisfied in the solution. This yields still another equation

$$H(\gamma_3, \gamma_4, c, \mu) = 0. \tag{54}$$

For $\mu = 0$, the problem has a solution depending on two arbitrary constants ω and h; thus, for $\mu = 0$,

$$\left.\begin{aligned} x_1(\sigma, 0, 0, \gamma_3^0, \gamma_4^0, c_0, 0) \equiv 0, \\ x_2(\sigma, 0, 0, \gamma_3^0, \gamma_4^0, c_0, 0) \equiv 0. \end{aligned}\right\} \quad (55)$$

And, in consequence of this,

$$\left.\frac{\partial x_1(\sigma)}{\partial \gamma_j}\right|_{\mu=0} = \left.\frac{\partial x_2(\sigma)}{\partial \gamma_j}\right|_{\mu=0} \equiv 0, \quad j = 1, 2. \quad (56)$$

In this case, $c_0 = \frac{4\sigma^2}{\pi^2 k^2}$ $k = 0, 1, 2, \ldots$. Equation $H = 0$, for $\mu = 0$, will take the form

$$\frac{1}{2}\left(\gamma_3^{0^2} + \gamma_4^{0^2}\right) - 4 = \frac{h}{2} = 0. \quad (57)$$

Let us demonstrate that, for sufficiently small values of μ, eqs.(53), (54) when fulfilling certain conditions will determine the functions γ_3, γ_4, c analytic with respect to μ. Let $\beta_1 = \gamma_3 - \gamma_3^{(0)}$, $\beta_2 = \gamma_4 - \gamma_4^{(0)}$, $c_1 = c - c_0$. Then, the equations $x_1(\sigma) = x_2(\sigma) = 0$ can be represented in the form

$$\left.\begin{aligned} x_1(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = \left(\frac{\partial x_1}{\partial \gamma_3} + U_1\right)\beta_1 + \left(\frac{\partial x_1}{\partial \gamma_4} + U_2\right)\beta_2 + \left(\frac{\partial x_1}{\partial c} + U_3\right)c_1 + \left(\frac{\partial x_1}{\partial \mu} + U_4\right)\mu, \\ x_2(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = \left(\frac{\partial x_2}{\partial \gamma_3} + V_1\right)\beta_1 + \left(\frac{\partial x_2}{\partial \gamma_4} + V_2\right)\beta_2 + \left(\frac{\partial x_2}{\partial c} + V_3\right)c_1 + \left(\frac{\partial x_2}{\partial \mu} + V_4\right)\mu, \end{aligned}\right\} \quad (58)$$

where u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 are analytic functions of β_1, β_2, c, μ which vanish for $\mu = \beta_1 = \beta_2 = c_1 = 0$; the derivatives are calculated for $\mu = 0$, $c = c_0$.

We will divide eqs.(58) by μ and make μ tend to zero. Assuming $\nu = \lim \frac{c_1}{\mu}$ and taking into account eq.(56), we obtain

$$\left.\begin{aligned} \Phi_1\left(\gamma_3^{(0)}, \gamma_4^{(0)}, \nu\right) = \left.\frac{\partial x_1(\sigma)}{\partial c}\right|_{\mu=0} \nu + \left.\frac{\partial_1 x(\sigma)}{\partial \mu}\right|_{\mu=0} = 0, \\ \Phi_2\left(\gamma_3^{(0)}, \gamma_4^{(0)}, \nu\right) = \left.\frac{\partial x_2(\sigma)}{\partial c}\right|_{\mu=0} \nu + \left.\frac{\partial x_2(\sigma)}{\partial \mu}\right|_{\mu=0} = 0. \end{aligned}\right\} \quad (59)$$

To these equations we add

$$\Phi_3 = \gamma_3^{0^2} + \gamma_4^{0^2} - 8 = h = 0.$$

If γ_3^0, γ_4^0, ν satisfy these equations and if $\frac{P(\Phi_1, \Phi_2, \Phi_3)}{P(\gamma_3^0, \gamma_4^0, \nu)} \neq 0$, then eqs.(53), (54) actually determine, for small μ, the functions γ_3, γ_4, c analytic with respect to μ. In this case, $\gamma_3(0) = \gamma_3^0$, $\gamma_4(0) = \gamma_4^{(0)}$, $\left.\frac{dc}{d\mu}\right|_{\mu=0} = \nu$. It remains

to prove the equivalence of the obtained conditions with the previously derived conditions (36), (37). We note that $x_1(s)$ and $x_2(s)$ satisfy eqs.(7). Let /164
us then differentiate these equations with respect to μ, for $\mu = 0$,

$$\frac{d}{ds}\frac{\partial x_j}{\partial \mu}\bigg|_{\mu=0} = \sum_{i=0}^{4}\frac{\partial X_j}{\partial x_i}\frac{\partial x_i}{\partial \mu}\bigg|_{\mu=0} + F_j^{(1)}\bigg|_{\mu=0}\left(F_j^{(1)} = \frac{\partial X_j}{\partial \mu}\right). \tag{60}$$

Furthermore, from eq.(52), if we differentiate with respect to μ, we obtain

$$\frac{\partial x_j^{(0)}}{\partial \mu}\bigg|_{\mu=0} = 0.$$

Consequently, it is necessary to find the particular solution of eqs.(60) with zero initial data. As was shown, such a solution has the form

$$\frac{\partial x_1(s)}{\partial \mu}\bigg|_{\mu=0} = -\frac{1}{4\sqrt{a}}\left(x_2 \cot 2ans + \frac{2}{n}\dot{x}_1 s - \frac{x_2}{n}t\right)\int_0^s (x_2 F_j^{(1)} - x_1 F_4^{(1)})\,ds +$$

$$+\frac{1}{4\sqrt{a}}\dot{x}_2\int_0^s \left\{\left[x_2 \cot 2ans + \frac{2}{n}\dot{x}_1 s - \frac{x_2}{n}t\right]F_3^{(1)} + \left[-x_1 \cot 2ans + \frac{2}{n}\dot{x}_2 s + \frac{x_1}{n}t\right]F_4^{(1)}\right\}ds +$$

$$+\frac{1}{8}(x_1 + t x_2)\int_0^s [\dot{x}_1 F_3^{(1)} + \dot{x}_2 F_4^{(1)}]\,dx - \frac{\dot{x}_1}{8}\int_0^s [(x + t x_2)F_3^{(1)} + (x_2 - t x_1)F_4^{(1)}]\,ds$$

and the analogous expression for $\frac{\partial x_2}{\partial \mu}\Big|_{\mu=0}$. The expressions for $\frac{\partial x_3}{\partial \mu}\Big|_{\mu=0}$ and $\frac{\partial x_4}{\partial \mu}\Big|_{\mu=0}$ are not needed. Setting $s = \sigma$, after easy calculations, we have

$$\left.\begin{aligned}
\frac{\partial x_1(\sigma)}{\partial \mu}\bigg|_{\mu=0} &= \frac{\sqrt{2}}{4}[\cos\alpha + \tau\sin\alpha]\,I_1 + \frac{1}{\sqrt{8}}I_2\sin\alpha,\\
\frac{\partial x_2(\sigma)}{\partial \mu}\bigg|_{\mu=0} &= \frac{\sqrt{2}}{4}[-\sin\alpha + \tau\cos\alpha]\,I_1 + \frac{1}{\sqrt{8}}I_2\cos\alpha,
\end{aligned}\right\} \tag{61}$$

where

$$\left.\begin{aligned}
&\alpha = \frac{2\pi}{n}k + \omega,\\
&I_1 = \int_0^\sigma [x_2 F_3^{(1)} - x_1 F_4^{(1)}]\,dx, \quad I_2 = \int_0^\sigma [(x_1 + tx)F_3^{(1)} + (x_2 - tx_1)F_4^{(1)}]\,dx,
\end{aligned}\right\} \tag{62}$$

The quantities $\frac{\partial x_1(\sigma)}{\partial c}$ are calculated directly:

$$\frac{\partial x_1(\sigma)}{\partial c}\bigg|_{\mu=0} = -\frac{\sqrt{2}}{4}\tau\sin\alpha, \quad \frac{\partial x_2(\sigma)}{\partial c} = -\frac{\sqrt{2}}{4}\tau\cos\alpha.$$

Equations (59) are written in the form

$$\Phi_1 = -\frac{\sqrt{2}}{4}\tau\nu\sin\alpha + \frac{\sqrt{2}}{4}(\cos\alpha + \tau\sin\alpha)\dot{I}_1 + \frac{\sin\alpha}{\sqrt{8}}I_2 = 0,$$

$$\Phi_2=-\frac{\sqrt{2}}{4}\tau\nu\cos\alpha+\frac{\sqrt{2}}{4}(-\sin\alpha+\tau\cos\alpha)I_1+\frac{\cos\alpha}{\sqrt{8}}I_2=0.$$

Let us multiply the first equation by cos α, the second by sin α and then subtract one from the other. This yields

$$\frac{\sqrt{2}}{4}I_1=0. \tag{63}$$

Having multiplied the first by sin α, the second by cos α, and having added both, we have /165

$$-\frac{\sqrt{2}}{4}\tau\nu+\frac{\sqrt{2}}{4}\tau I_1+\frac{1}{\sqrt{8}}I_2=\frac{1}{\sqrt{8}}(-\tau\nu+I_2)=0. \tag{64}$$

Calculating the value of I_1 and I_2 and comparing with eqs.(36), (37), we note that

$$I_1[\gamma_3^0(\omega,h),\ \gamma_4^{(0)}(\omega,h)]=P_1(\omega,h),$$
$$I_2[\gamma_3^0(\omega,h),\ \gamma_4^{(0)}(\omega,h)]-\tau\nu=P_2(\omega,h,\nu).$$

It remains to calculate $\dfrac{D(\Phi_1,\ \Phi_2,\ \Phi_3)}{D(\gamma_3^{(0)},\ \gamma_4^{(0)},\ \nu)}$. Using the equalities (63), (64), we have

$$\frac{D(\Phi_1,\Phi_2,\Phi_3)}{D(\gamma_3^{(0)},\gamma_4^{(0)},\nu)}=\frac{D(\Phi_1,\Phi_2,\Phi_3)}{D(P_1,P_2,\Phi_3)}\cdot\frac{D(P_1,P_2,\Phi_3)}{D(\omega,h,\nu)}\cdot\frac{D(\omega,h,\nu)}{D(\gamma_3^{(0)},\gamma_4^{(0)},\nu)},$$

$$\frac{D(\Phi_1\ \Phi_2,\Phi_3)}{D(P_1,P_2,\Phi_3)}=\begin{vmatrix}\frac{\sqrt{2}}{4}\cos\alpha+\frac{\sqrt{2}}{4}\tau\sin\alpha, & \frac{\sin\alpha}{\sqrt{8}}, & 0\\ -\frac{\sqrt{2}}{4}\sin\alpha+\frac{\sqrt{2}}{4}\tau\cos\alpha, & \frac{\cos\alpha}{\sqrt{8}}, & 0\\ 0, & 0, & 1\end{vmatrix}=\frac{1}{8},$$

$$\frac{D(P_1,P_2,\Phi_3)}{D(\omega,h,\nu)}=\begin{vmatrix}\frac{\partial P_1}{\partial\omega}, & \frac{\partial P_1}{\partial h}, & 0\\ \frac{\partial P_2}{\partial\omega}, & \frac{\partial P_2}{\partial h}, & -\tau\\ 0, & 1, & 0\end{vmatrix}=\tau\frac{\partial P_1}{\partial\omega},$$

$$\frac{D(\omega,h,\nu)}{D(\gamma_3^0,\gamma_4^0,\nu)}=\frac{1}{\dfrac{D(\gamma_3^{(0)},\gamma_4^{(0)},\nu)}{D(\omega,h,\nu)}}=\frac{1}{\begin{vmatrix}-\sqrt{2a}\,2an\sigma\cos\alpha, & -\frac{\sqrt{a}}{8\sqrt{2}}\sin\alpha\cdot 2an\sigma, & 0\\ \sqrt{2a}\,2an\sigma\sin\alpha, & -\frac{\sqrt{a}}{8\sqrt{2}}\cos\alpha\cdot 2an\sigma, & 0\\ 0, & 0 & 1\end{vmatrix}}=\frac{2}{\sigma^2}.$$

Hence,

$$\frac{D(\Phi_1,\Phi_2,\Phi_3)}{D(\gamma_3^0,\gamma_4^0,\nu)}=\frac{\tau}{4\sigma^2}\left[\frac{\partial P_1(\omega,0)}{\partial\omega}\right]_{\omega=\omega^*}\neq 0.$$

Thus, the conditions (38), (39), which are necessary and sufficient for constructing the formal series, are at the same time necessary and sufficient conditions for the existence of the solution to the problem.

Since the coefficients of the formal series are determined uniquely, it follows that these series converge.

5. Investigation of the Fundamental Equation (38)

Let us examine the fundamental equation (38)

$$\int_0^\tau y\left(1-\frac{1}{r_2^3}\right)dt=0.$$

This equation can be rewritten in the following form:

$$I(\omega)=\int_0^\tau r(t)\sin(\omega+t)\left(1-\frac{1}{\sqrt{[1+r^2(t)-2r(t)\cos(\omega+t)]^3}}\right)dt=0,$$

where /166

$$r(t)=a(1-\cos E),\ t=\frac{E-\sin E}{n},\ \ \tau=\frac{2\pi}{n}k\ \ k=1,\ 2,\ 3...$$

It is obvious that r(t) is symmetric with respect to the middle of the interval [0, τ]. If we select ω such that $\int_0^\tau \sin(\omega+t)dt=0$, then the function cos (ω + t) will also be symmetric with respect to the point $\frac{\tau}{2}$. It follows that, for such values of ω, I(ω) = 0. The generating solution corresponding to these values will be symmetric with respect to the x-axis.

Thus, we have two trivial roots $\omega_1=-\frac{\tau}{2}$, $\omega_2=\pi-\frac{\tau}{2}$. The corresponding path is plotted in Fig.1.

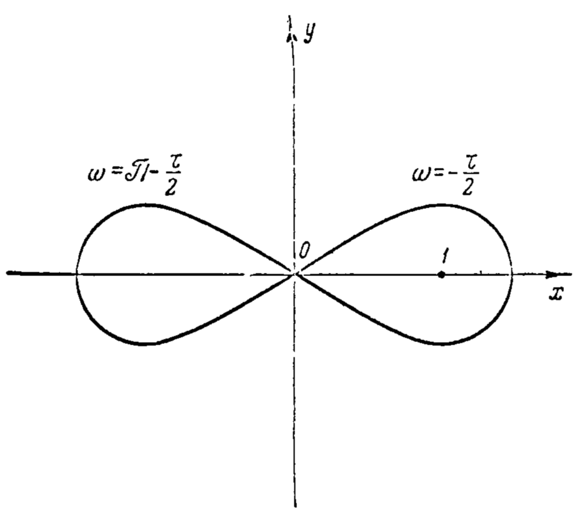

Fig.1 Symmetric Orbits.

The paths were obtained by Levi-Civita (1904) by a different method.

It is easy to prove the existence of solutions, asymmetric with respect to the x-axis. In fact, let $\tau = \frac{2\pi}{n} k$ be small; in this case a will also be small. It is obvious that the curve $x = r(t) \cos (\omega + t)$, $y(t) = -r(t) \sin (\omega + t)$ lies entirely within the angle made by tangents to this curve if $t = 0$ and $t = \tau = \frac{2\pi}{n} k$. The osculating point of the slope of these tangents, for $t = 0$ and $t = \tau$, is $-\tan \omega$ and $-\tan (\omega + \tau)$, respectively. We will set $\omega = 0$. Then, for sufficiently small values of τ, the curve $x = r(t) \cos (\omega + t)$, $y = -r(t) \sin (\omega + t)$ lies entirely within the lower half-plane within a circle of unit radius with a center at (1, 0). Consequently, $I(0) < 0$. Having set $\omega = \pi/2$, we find that the corresponding curve also lies within the lower half-plane but outside the indicated circle, i.e., $I\left(\frac{\pi}{2}\right) > 0$.

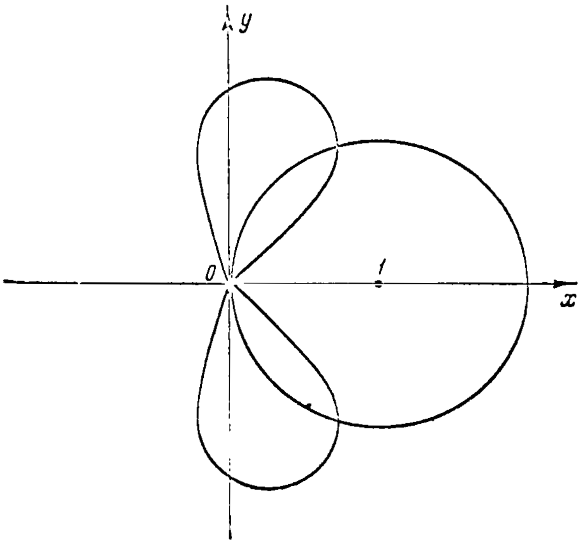

Fig.2 Orbits of the Second Kind.

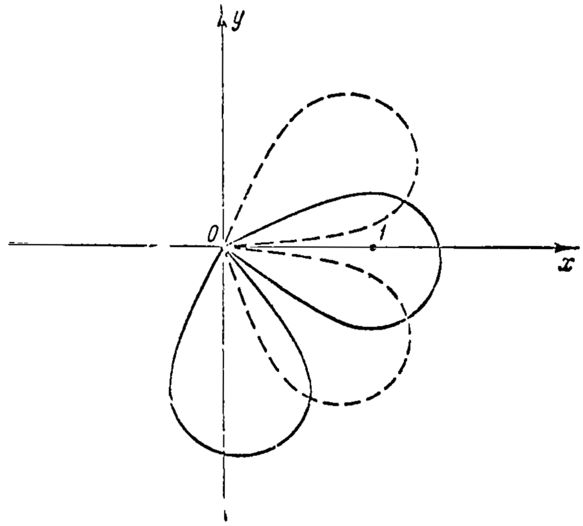

Fig.3 Orbits of the Third Kind. The broken line gives the symmetric orbit and the solid line, the orbit of the third kind.

Thus there exists a root of the equation $I(\omega) = 0$ located in the interval from 0 to $\frac{\pi}{2}$; since eq.(3) does not change when replacing y by -y, it follows that, on the curve symmetric to that obtained with respect to the x-axis, $I(\omega)$ will also be equal to zero. Figure 2 gives an example of the corresponding paths. /167

Finally, we can prove the existence of still another, more interesting, type of solution. Let $\tau = \frac{2\pi}{n} k$ and $k \geq 2$, $a > 0.5$. Equation $I(\omega) = 0$ can then be rewritten as

$$I(\omega) = \frac{\partial \Phi(\omega)}{\partial \omega} = 0,$$

where

$$\Phi = \int_0^{\tau} \left(-x + \frac{1}{r_2}\right) dt.$$

Since $a > 0.5$, a change of ω will cause the function $\Phi(\omega)$ to become infinite for certain values of ω. Consequently, between two such successive values the function $\Phi(\omega)$ has at least one minimum, i.e., $I(\omega) = 0$. Generally speaking, the corresponding curves, for $k > 1$, are not symmetric (Fig.3). In fact, if $a \to 0.5$ from above, it is obvious that the two successive values of ω_1 and ω_2, for which $\Phi(\omega) = \infty$, tend to a common limit. This limit is determined from the conditions: $r(t, \omega) = 0$, $r_2(t, \omega) = 0$, $a = 0.5$ and is equal to $-\frac{\pi}{n}$. Since the root of the equation $I(\omega) = 0$ lies between ω_1 and ω_2, then this root also tends to $-\frac{\pi}{n}$ as $a \to 0.5$; thus, for $k \neq 1$, it does not equal the roots corresponding to the symmetric solutions.

In conclusion I wish to thank G.A.Merman, under whose supervision this paper was written.

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Translated for the National Aeronautics and Space Administration by the O.W.Leibiger Research Laboratories, Inc.

NASA-Langley, 1967 —— 30 F-464

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